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MODELLING OF FLEXIBLE SURFACES: A PRELIMINARY STUDY

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Abstract—We give a careful derivation of the 1-dimensional classical scalar "string" equation which involves linearization about a horizontal reference or equilibrium position. We then derive a model for "small motion" about a nonhorizontal reference. The implications of our findings to modelling of flexible antenna surfaces such as that in the Maypole Hoop/Column antenna are discussed.

1. INTRODUCTION

The investigations reported herein are motivated by our interest in equations governing the antenna surface in large space antennas such as the Maypole Hoop/Column configuration depicted in Fig. 1.

This antenna consists of a gold-plated molybdenum reflective mesh surface stretched over a collapsible hoop that supplies the rigidity necessary to maintain the outer circular shape of the antenna. Of fundamental interest in estimation and control of the antenna are accurate models for the flexible membrane-like mesh surface. Important modelling questions include whether one can use a simple scalar "membrane" equation or must use a vector system. Also, can one use a linear equation or system of equations, or are nonlinear equations necessary to describe the shape (dynamic or static) of the surface?

In this preliminary study, we won't provide conclusive answers to these questions. But we shall establish some rather pertinent results that suggest what the answers will be when final models are derived. Our approach here is as follows. Rather than attempt a full 3-dimensional model for the surface, we analyze carefully a 1-dimensional flexible "membrane"—i.e., a string. One might view this "string" as a section of the antenna surface obtained by passing a vertical plane through the antenna.

We use first principles to write the basic nonlinear equations for a string. We then give a careful derivation of a linearization about a horizontal equilibrium or reference configuration and point out clearly how one arrives at the familiar single scalar wave equation. We next consider linearization about a curved equilibrium. These findings are important with regard to a definitive investigation of a multi-dimensional model for the Maypole Hoop/Column surface. Our conclusions are summarized in the last section of this report.

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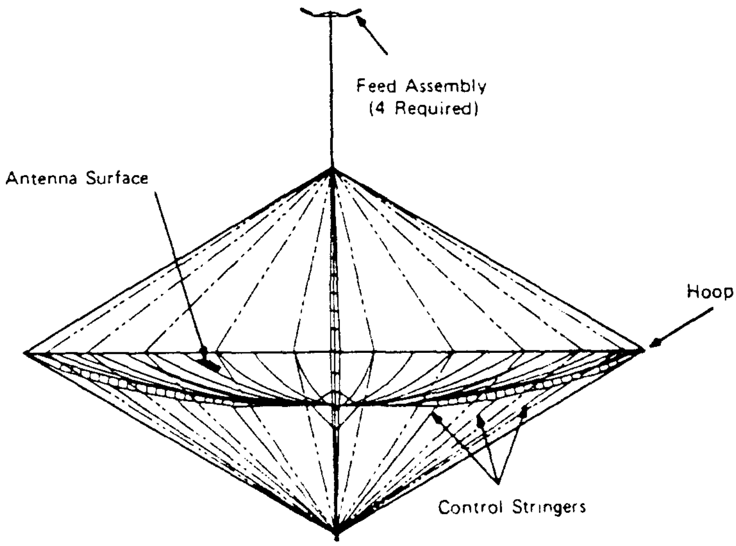

Fig. 1. Maypole Hoop/Column configuration.

2. BASIC EQUATIONS GOVERNING AN ELASTIC STRING IN R^2

In this section we formulate the initial-boundary value problem governing the two-dimensional motion of an elastic string about its equilibrium position. We shall derive the differential equation in *strong form*, assuming smoothness of functions as needed. Our presentation, based on the principle of conservation of linear momentum, follows classical arguments such as those found in Weinberger [4] and Antman [1]. Following these references, we state carefully our assumptions about the string and its configuration. In the latter part of our derivation, we shall distinguish between two cases: a string with horizontal equilibrium and a string with curved equilibrium. The differences are reflected in our formulation for the tension in the string.

A somewhat analogous derivation under less stringent smoothness can be carried out using energy considerations or the Impulse-Momentum Law (a conservation principle)—see [1]. Such an approach yields the state equations in *weak form* and since these will be of interest in our estimation research for the Maypole Hoop/Column antenna, a two-dimensional analogue (for a stretched membrane) will be considered elsewhere. For the present, we consider here only derivations that lead to models with equations in strong form.

We consider a string stretched between two pegs (at $(0, 0)$ and $(l, 0)$ in the (x, y) plane), possibly subject to external forces. We suppose the string has a nowhere vertical equilibrium position described by the curve $E(s) = (x_E(s), y_E(s)) = (s, h(s))$, $0 \le s \le l$. That is, we let s denote the x-coordinate of a particular material point (molecule) when the string is in its equilibrium position. We note that each material particle of the string is thus uniquely labeled since in its equilibrium position the string is nowhere vertical. We first assume

HYPOTHESIS 2.1. The string is so thin that its cross section moves as a single point; this movement is restricted to the (x, y) plane.

If the string satisfies (H2.1), then the motion of the string can be described by giving at each time t, the *displacement* or *position* vector $r(s, t) = (x(s, t), y(s, t))$ of each material point s. We further assume that

HYPOTHESIS 2.2. We may regard the string as a continuum, ignoring the fact that it is composed of individual molecules. The string has a continuous linear mass density ρ such that for any segment (s_1, s_2), $0 \le s_1 \le s_2 \le l$, the mass is given by $\int_{s_1}^{s_2} \rho(s)\mathrm{d}s$.

For each $t \ge 0$ and $0 < s < l$, let $\tau^-(s, t)$ denote the vector force exerted on the material segment $[0, s)$ by the material segment $[s, l]$. Similarly, let $\tau^-(s, t)$ denote the force exerted on the segment $(s, l]$ by the segment $[0, s]$. Then the resultant forces on a segment $[s_1, s_2]$ due to the remainder of the string is $\tau^-(s_2, t) + \tau^-(s_1, t)$. Letting $F(r, t) = (F_1(r, t), F_2(r, t))$ denote the net external force per unit density exerted on the string, we find that the net external force on a segment (s_1, s_2) is given by $\int_{s_1}^{s_2} \rho(s)F(r, t)\mathrm{d}s$. The *principle of conservation of linear momentum* may then be used to write the balance equations for a segment (s_1, s_2)

$$\frac{\partial}{\partial t}\int_{s_1}^{s_2} \rho(s)\frac{\partial r}{\partial t}(s, t)\mathrm{d}s = \tau^-(s_2, t) + \tau^-(s_1, t) + \int_{s_1}^{s_2} \rho(s)F(r, t)\mathrm{d}s.$$

Continuity arguments (assuming that $\partial^2 r/\partial t^2$ and F are continuous) can be used to argue $\tau^- = -\tau^+$. Using this and denoting τ^+ by just τ, we arrive at the equation of motion for a segment (s_1, s_2)

$$\int_{s_1}^{s_2} \rho(s)\frac{\partial^2 r}{\partial t^2}(s, t)\mathrm{d}s = \tau(s_2, t) - \tau(s_1, t) + \int_{s_1}^{s_2} \rho(s)F(r, t)\mathrm{d}s. \tag{2.1}$$

If we further assume that τ is continuously differentiable, we may differentiate (2.1) with respect tof s_2 to obtain

$$\rho(s)\frac{\partial^2 r}{\partial t^2}(s, t) = \frac{\partial \tau}{\partial s}(s, t) + \rho(s)F(r, t). \tag{2.2}$$

Turning to the force term τ we make the following assumption, which is the *defining property of a string.*

HYPOTHESIS 2.3. The force $\tau(s, t)$ acts in a direction tangent to the displacement vector $r(s, t)$ with amplitude $T(s, t)$ called the *tension.*

Since r is the displacement vector, the unit tangent vector to the string at each point is given by $(\partial r/\partial s)/\,|\,(\partial r/\partial s)\,|$ and hence we have

$$\tau(s, t) = \frac{T(s, t)}{\sqrt{\left(\frac{\partial x}{\partial s}\right)^2 + \left(\frac{\partial y}{\partial s}\right)^2}}\left(\frac{\partial x}{\partial s}, \frac{\partial y}{\partial s}\right). \tag{2.3}$$

Using this representation of τ, we may write (2.2) component-wise as

$$\rho(s)\frac{\partial^2 x}{\partial t^2} = \frac{\partial}{\partial s}\left[\frac{T(s, t)}{\sqrt{\left(\frac{\partial x}{\partial s}\right)^2 + \left(\frac{\partial y}{\partial s}\right)^2}}\frac{\partial x}{\partial s}\right] + \rho(s)F_1(x, y, t) \tag{2.4}$$

$$\rho(s)\frac{\partial^2 y}{\partial t^2} = \frac{\partial}{\partial s}\left[\frac{T(s, t)}{\sqrt{\left(\frac{\partial x}{\partial s}\right)^2 + \left(\frac{\partial y}{\partial s}\right)^2}}\frac{\partial y}{\partial s}\right] + \rho(s)F_2(x, y, t)\,. \tag{2.5}$$

In order to use these equations, we must further specify the tension $T(s, t)$. This will be related to the equilibrium or reference position and before discussing this, let us complete the formulation of our initial-boundary value problem. We assume that in addition to (2.4), (2.5), we have given functions f_1, f_2, g_1, g_2 specifying the *initial conditions*

$$\left.\begin{aligned} r(s, 0) &= (f_1(s), f_2(s)) \\ \frac{\partial r}{\partial t}(s, 0) &= (g_1(s), g_2(s)) \end{aligned}\right\}, \quad 0 \leq s \leq l. \tag{2.6}$$

We assume that the ends of the string remain fixed so that the *boundary conditions* are given by

$$r(0, t) = (0, 0), \qquad r(l, t) = (l, 0), \qquad t \geq 0. \tag{2.7}$$

For compatibility between the initial conditions and boundary conditions we must, of course, have $f_i(0) = g_i(0) = 0$, $i = 1, 2$, and $f_1(l) = l$, $f_2(l) = 0$.

We return now to the tension $T(s, t)$, a specification of which involves assumptions on the material properties of the string as well as its equilibrium position. We first hypothesize that the string is *perfectly elastic*, i.e.,

HYPOTHESIS 2.4. The tension at any material point located at $r(s, t)$ is determined by the elongation or local stretching per unit length of the string with respect to its equilibrium position.

The local stretching per unit length of the string at any time is given by

$$\begin{aligned} \left|\frac{\partial r}{\partial s}\right| = \lim_{\Delta s \to 0}\left|\frac{\Delta r}{\Delta s}\right| &= \lim_{\Delta s \to 0}\frac{\sqrt{(\Delta x)^2 + (\Delta y)^2}}{\Delta s} \\ &= \sqrt{\left(\frac{\partial x}{\partial s}(s, t)\right)^2 + \left(\frac{\partial y}{\partial s}(s, t)\right)^2} \end{aligned} \tag{2.8}$$

whereas the local stretching per unit length with respect to its equilibrium is given by

$$\begin{aligned} e(s, t) &= \lim_{\Delta s \to 0}\left\{\frac{|\Delta r| - |\Delta r_E|}{\Delta s}\right\} \\ &= \sqrt{\left(\frac{\partial x}{\partial s}\right)^2 + \left(\frac{\partial y}{\partial s}\right)^2} - \sqrt{\left(\frac{\partial x_E}{\partial s}\right)^2 + \left(\frac{\partial y_E}{\partial s}\right)^2} \end{aligned} \tag{2.9}$$

where $E(s) = (x_E(s), y_E(s))$ is the equilibrium position for the string.

Consider first the case of an elastic string with *horizontal equilibrium*. In this case, the variable s actually measures arc length since $E(s) = (s, 0)$ and the elongation in (2.9) reduces to

$$e(s, t) = \sqrt{\left(\frac{\partial x}{\partial s}\right)^2 + \left(\frac{\partial y}{\partial s}\right)^2} - 1. \tag{2.10}$$

For the more general case of a nowhere vertical nonhorizontal or *curved equilibrium* given by $E(s) = (s, h(s))$ we find

$$e(s, t) = \sqrt{\left(\frac{\partial x}{\partial s}\right)^2 + \left(\frac{\partial y}{\partial s}\right)^2} - \sqrt{1 + (h'(s))^2}. \tag{2.11}$$

In either case, the hypothesis (H2.4) implies the existence of a function T that relates the tension $T(s, t)$ to $e(s, t)$. In this case, we assume that T also depends on the particular material point s (i.e., we do not require uniform "stiffness" in the string—a generality that is of some importance in our investigations of nonuniform structures such as antenna surfaces with webbing to partially distribute the loads). Thus we have

$$T(s, t) = \mathrm{T}(e(s, t), s) \tag{2.12}$$

where the value $\mathrm{T}(e, s)$ describes the elastic property of the string with elongation e at s.

In summary, our string satisfies the equations of motion

$$\rho(s)x_{tt} = \frac{\partial}{\partial s}\left[\frac{\mathrm{T}(e(s, t), s)}{\sqrt{x_s^2 + y_s^2}}\frac{\partial x}{\partial s}\right] + \rho(s)F_1(x, y, t), \tag{2.13}$$

$$\rho(s)y_{tt} = \frac{\partial}{\partial s}\left[\frac{\mathrm{T}(e(s, t), s)}{\sqrt{x_s^2 + y_s^2}}\frac{\partial y}{\partial s}\right] + \rho(s)F_2(x, y, t), \tag{2.14}$$

where e is given by (2.10) or (2.11). Appropriate initial and boundary conditions are given by (2.6), (2.7).

In the next section we shall turn to a linearization theory for this system of nonlinear equations. For further comments on these equations themselves, see [1, 2, 3].

3. LINEARIZATION ABOUT A HORIZONTAL EQUILIBRIUM

The nonlinear system (2.13, 2.14) derived in the previous section is difficult to solve whether one considers horizontal or curved equilibrium positions as reference in determining the tension. In this section we show how, in the case of a horizontal equilibrium, one can approximate this system by a *scalar linear* equation for small amplitude, essentially transverse (i.e., in the y-direction) motions of the string. This approximation is formal in the sense that while we state precisely our approximation hypotheses, we shall not argue that solutions to our linear equation do indeed approximate solutions of the nonlinear system (2.13, 2.14).

We consider linearization about a horizontal equilibrium $E(s) = (x_E(s), y_E(s)) = (s, 0)$. We assume that ρ is continuous and both T and F are twice continuously differentiable in all arguments. We further assume:

HYPOTHESIS 3.1. The net external force in the x direction acts at any time simply to restore the string to its equilibrium position.

We observe that this assumption implies in particular that $F_1(x_E(s), y, t) \equiv 0$ for $0 \le s \le l$, all y and t, or in this case $F_1(x, y, t) \equiv 0$. Our fundamental *small amplitude assumption* is embodied as:

HYPOTHESIS 3.2. The motion of the string consists of small movements about the equilibrium E. Specifically, the motion can be described by

$$x(s, t) = s + \phi(s, t) \tag{3.1}$$

$$y(s, t) = \eta(s, t) \tag{3.2}$$

where ϕ, η and their derivates are small.

We further assume that ϕ is small relative to η, or that the motion is *essentially transverse*. More precisely, we hypothesize

HYPOTHESIS 3.3. For all $t \geq 0$ and $0 \leq s \leq l$, we have $|\phi(s, t)| \ll |\eta(s, t)|$ and furthermore both ϕ and all its derivatives are negligibly small.

Finally, we also assume

HYPOTHESIS 3.4. The string is never vertical so that $(\partial x/\partial s) \neq 0$ for $0 \leq s \leq l$, $t \geq 0$.

The implication of this last hypothesis is that we may invoke the implicit function theorem with (3.1) to solve for $s = s(x, t)$. We can therefore express η of (3.2) as a function of (x, t) which we shall denote hereafter as v. That is, we have

$$v(x, t) = v(x(s, t), t) = \eta(s(x, t), t) = \eta(s, t). \tag{3.3}$$

In connection with this new function for the transverse motion, we make one final assumption which will be needed in the *linearization* below.

HYPOTHESIS 3.5. The function v given in (3.3) satisfies $|\partial v/\partial x| < 1$ for $0 \leq x \leq l$ and $t \geq 0$.

Before proceeding with our derivation, we note that while (H3.4) and the use of the implicit function theorem in defining v may appear rather pedantic as opposed to assuming $\phi = 0$ so that $s = x$, this latter assumption is certainly less desirable from a physical viewpoint.

Returning to consider (2.13) and (2.14), we shall continue our deliberations under the following *linearization principles*:

I (Small motion): We neglect terms that are quadratic or of higher degree in ϕ, η, and their derivatives.

II (Transverse motion): We neglect terms which are linear in ϕ and its derivatives.

From (3.1) and (3.2), we have at once that $x_{tt} = \phi_{tt}$ and $y_{tt} = \eta_{tt}$. But since $F_1 \equiv 0$ and ϕ_{tt} is negligible, we see that (2.13) itself only involves negligible terms and thus can be ignored in a linearization. So we consider the terms in (2.14). Since F_2 is C^2, we expand it about $(x, 0)$ using Taylor's theorem to obtain

$$F_2(x, y, t) = F_2(x, 0, t) + \frac{\partial F_2}{\partial y}(x, 0, t)\eta + O(\eta^2). \tag{3.4}$$

If, furthermore, we assume ρ is C^1, we have

$$\rho(s) = \rho(x) + O(\phi).$$

We thus may rewrite (2.14) as

$$\rho(x)\eta_{tt} = \frac{\partial}{\partial s}\left[\frac{\mathrm{T}(e,s)}{\sqrt{x_s^2 + y_s^2}}\frac{\partial \eta}{\partial s}\right] + \rho(x)\{F_2(x,0,t) + \frac{\partial F_2}{\partial y}(x,0,t)\eta\} + O(\phi + \eta^2). \quad (3.5)$$

Using the chain rule and (3.3), we find

$$\begin{aligned}
\frac{\partial \eta}{\partial s} &= \frac{\partial v}{\partial x}\frac{\partial x}{\partial s} \\
\frac{\partial \eta}{\partial t} &= \frac{\partial v}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial v}{\partial t} \\
\frac{\partial^2 \eta}{\partial t^2} &= \frac{\partial^2 v}{\partial x^2}\left(\frac{\partial x}{\partial t}\right)^2 + 2\frac{\partial^2 v}{\partial x \partial t}\frac{\partial x}{\partial t} + \frac{\partial v}{\partial x}\frac{\partial^2 x}{\partial t^2} + \frac{\partial^2 v}{\partial t^2}.
\end{aligned} \quad (3.6)$$

But since $x = s + \phi$, this last equation can be written

$$\frac{\partial^2 \eta}{\partial t^2} = \frac{\partial^2 v}{\partial t^2} + O(\phi_t^2 + \phi_t + \phi_{tt}). \quad (3.7)$$

Futhermore, the first term in the right side of (3.5) is the same as (using (3.6))

$$\frac{\partial}{\partial x}\left[\frac{\mathrm{T}(e,s)}{\sqrt{1 + v_x^2}}\frac{\partial v}{\partial x}\right]\{1 + \phi_s\}$$

which, in view of our assumptions, will be approximated by

$$\frac{\partial}{\partial x}\left[\frac{\mathrm{T}(e,s)}{\sqrt{1 + v_x^2}}\frac{\partial v}{\partial x}\right].$$

Finally, we turn to an approximation of $\mathrm{T}(e, s)$. First, let $\bar{e}$ denote the elongation e expressed in terms of (x, t). That is,

$$\bar{e}(x,t) = e(s,t) = \frac{\partial x}{\partial s}\sqrt{1 + v_x^2} - 1.$$

Using the fact that $\partial x/\partial s = 1 + \partial\phi/\partial s$ and $(1 + v_x^2)^{1/2} = 1 + (v_x^2/2) - \frac{1}{4}v_x^4 + \cdots$, we find

$$\bar{e}(x,t) = \phi_s + \frac{v_x^2}{2} - \frac{1}{4}v_x^4 + \cdots \approx \frac{v_x^2}{2} \quad (3.8)$$

where we have used (H3.3) and (H3.5) in making this approximation.

In a similar manner we arrive at the approximation

$$\frac{1}{\sqrt{1 + v_x^2}} \approx 1 - \frac{v_x^2}{2}, \quad (3.9)$$

while a Taylor's expansion for T yields ($s = x - \phi$)

$$\mathrm{T}(\tilde{e}, s) = \mathrm{T}(0, x) + \frac{\partial \mathrm{T}}{\partial e}(0, x)\tilde{e} + \frac{\partial \mathrm{T}}{\partial s}(0, x)\phi + \{\text{higher order terms}\}. \tag{3.10}$$

Combining (3.8), (3.9) and (3.10), we thus obtain

$$\begin{aligned}\frac{\partial}{\partial x}\left[\frac{\mathrm{T}(e, s)}{\sqrt{1 + v_x^2}}\frac{\partial v}{\partial x}\right] &\approx \frac{\partial}{\partial x}\left[\left\{\mathrm{T}(0, x) + \frac{\partial \mathrm{T}}{\partial e}(0, x)\frac{v_x^2}{2}\right\}\left(1 - \frac{v_x^2}{2}\right)\frac{\partial v}{\partial x}\right] \\ &\approx \frac{\partial}{\partial x}\left[\left(\mathrm{T}(0, x) + \left\{\frac{\partial \mathrm{T}}{\partial e}(0, x) - \mathrm{T}(0, x)\right\}\frac{v_x^2}{2}\right)\frac{\partial v}{\partial x}\right],\end{aligned}$$

where again we have used (H3.5) to drop terms of degree four or higher in v_x. Using this latter approximation with (3.7) in (3.5), we find the approximating nonlinear equation

$$\begin{aligned}\rho(x)v_{tt} = \frac{\partial}{\partial x}\left[\mathrm{T}(0, x)v_x + \left\{\frac{\partial \mathrm{T}}{\partial e}(0, x) - \mathrm{T}(0, x)\right\}\frac{v_x^3}{2}\right] \\ + \rho(x)\left[F_2(x, 0, t) + \frac{\partial F_2}{\partial y}(x, 0, t)v\right].\end{aligned} \tag{3.11}$$

Finally, if we use (H3.5) to neglect nonlinear terms we have our linear equation

$$\rho(x)v_{tt} = \frac{\partial}{\partial x}\left[\mathrm{T}(0, x)\frac{\partial v}{\partial x}\right] + \rho(x)\left[F_2(x, 0, t) + \frac{\partial F_2}{\partial y}(x, 0, t)v\right] \tag{3.12}$$

for *small amplitude transverse motion of a string* about a *horizontal equilibrium*. Recalling (2.6) and (2.7), we note that initial conditions

$$\begin{aligned} v(x, 0) &= f_2(x) \\ v_t(x, 0) &= g_2(x), \qquad 0 \le x \le l \end{aligned} \tag{3.13}$$

and boundary conditions

$$v(0, t) = v(l, t) = 0 \tag{3.14}$$

must be used in solving (3.12), where (H3.2) imposes a smallness assumption on f_2 and g_2 if one is expect the solution of (3.12) to approximate the actual motion of our string.

We note in passing that the above derivation reveals that the tension function T (see (3.8) and (3.10)) is an even function of the "strain" v_x (see [3] for an analysis in which this fact plays a role).

4. LINEARIZATION ABOUT A CURVED EQUILIBRIUM

In this section we consider a linearization of (2.13) and (2.14) to describe small amplitude motion about a curved equilibrium. We recall the notation for Section 2 where we defined a nonhorizontal, nonvertical equilibrium $E(s) = (x_E(s), y_E(s)) = (s, h(s))$ and defined the elongation (see (2.11)) by

$$e(s, t) = \sqrt{x_s^2 + y_s^2} - \sqrt{1 + h'(s)^2}. \tag{4.1}$$

We make the same smoothness assumption of Sections 2 and 3. For obvious reasons, analogues to (H3.1) and (H3.3) are *not* made. However, we do make an assumption of *small amplitude* motion analogous to (H3.2). Specifically we assume

HYPOTHESIS 4.1. The motion of the string consists of small movements about the equilibrium E. More precisely, the motion can be described by

$$x(s, t) = x_E(s) + \phi(s, t) = s + \phi(s, t) \tag{4.2}$$

$$y(s, t) = y_E(s) + \eta(s, t) = h(s) + \eta(s, t), \tag{4.3}$$

where ϕ, η, and their derivatives are small.

Our method consists of substituting (4.2) and (4.3) into (2.13) and (2.14), expanding various terms using Taylor's theorem and neglecting terms of higher order than first in ϕ, η and their derivatives. Thus, we again use the "small motion" *linearization principle I* stated in Section 3. Our basic linearization hypothesis (the analog of (H3.5)) will involve a "smallness" criterion on ϕ and η with respect to $dy_E/dx_E = dy_E/ds = h'$.

We proceed by expanding F_1, F_2 of (2.13) and (2.14). In view of (4.2) and (4.3) we have for $i = 1, 2$

$$F_i(x, y, t) \approx F_i(s, h(s), t) + \frac{\partial F_i}{\partial x}(s, h(s), t)\phi + \frac{\partial F_i}{\partial y}(s, h(s), t)\eta. \tag{4.4}$$

Furthermore, from (4.2) and (4.3) it follows that $x_{tt} = \phi_{tt}$, $y_{tt} = \eta_{tt}$ and it only remains to approximate the first terms in the right sides of (2.13) and (2.14). For this we shall invoke the *linearization* assumption:

HYPOTHESIS 4.2. The functions h, ϕ, and η of (4.2) and (4.3) satisfy

$$|[1 + h'(s)^2]^{-1}\{\phi_s^2 + \eta_s^2 + 2\phi_s + 2h'\eta_s\}| < 1.$$

We shall use this assumption to derive approximations for $e(s, t)$ and $(x_s^2 + y_s^2)^{-1/2}$. To do this, we make use of Taylor's expansions for $(1 + p)^{1/2}$ and $(1 + p)^{-1/2}$ where p is a quadratic in two variables. More precisely, suppose $p(r_1, r_2)$ is a quadratic polynomial in two real variables r_1, r_2 which satisfies $p(0, 0) = 0$. Then for (r_1, r_2) near $(0, 0)$ with $|p(r_1, r_2)| < 1$, we have

$$(1 + p(r_1, r_2))^{\pm 1/2} = 1 \pm \tfrac{1}{2}\{p_{r_1}(r_1, r_2)r_1 + p_{r_2}(r_1, r_2)r_2\} + 0(r_1^2 + r_2^2). \tag{4.5}$$

A simple application of (4.5) yields the following results: If A and B are two constants and $q(r_1, r_2) = A(r_1^2 + r_2^2 + 2r_1 + 2Br_2)$, then $q(0, 0) = 0$ and for all r_1, r_2 such that $|q(r_1, r_2)| < 1$, we have

$$(1 + q(r_1, r_2))^{\pm 1/2} = 1 \pm Ar_1 + ABr_2 + 0(r_1^2 + r_2^2). \tag{4.6}$$

We turn to e of (4.1) which, in vie of (4.2), (4.3) can be written

$$\begin{aligned} e(s, t) &= \sqrt{(1 + \phi_s)^2 + (h' + \quad)^2} - \sqrt{1 + (h')^2} \\ &= \sqrt{1 + (h')^2 + \phi_s^2 + \eta_s^2 + 2\phi_s + 2h'\eta_s} - \sqrt{1 + (h')^2} \\ &= \sqrt{1 + (h')^2}\,\{\sqrt{1 + (1 + (h')^2)^{-1}(\phi_s^2 + \eta_s^2 + 2\phi_s + 2h'\eta_s)} - 1\} \\ &= (1 + (h')^2)^{-1/2}\{\phi_s + h'\eta_s\} + O(\phi_s^2 + \eta_s^2), \end{aligned} \tag{4.7}$$

where we have used (4.6) with $A = (1 + (h')^2)^{-1}$, $B = h'$, $r_1 = \phi_s$, $r_2 = \eta_s$. In a similar manner, use of (4.6) yields,

$$\begin{aligned} (x_s^2 + y_s^2)^{-1/2} &= (1 + (h')^2)^{-1/2}\{1 + (1 + (h')^2)^{-1}(\phi_s^2 + \eta_s^2 + 2\phi_s + 2h'\eta_s)\}^{-1/2} \\ &= (1 + (h')^2)^{-1/2}(1 - (1 + (h')^2)^{-1}(\phi_s + h'\eta_s)) + O(\phi_s^2 + \eta_s^2). \end{aligned} \tag{4.8}$$

Finally, in view of (4.7) we have

$$\begin{aligned} \mathrm{T}(e, s) &= \mathrm{T}(0, s) + \frac{\partial \mathrm{T}}{\partial e}(0, s)e + O(e^2) \\ &= \mathrm{T}(0, s) + \frac{\partial \mathrm{T}}{\partial e}(0, s)(1 + (h')^2)^{-1/2}(\phi_s + h'\eta_s) + O(\phi_s^2 + \eta_s^2) \\ &= \mathrm{T}(0, s) + \frac{\partial \mathrm{T}}{\partial e}(0, s)\, K^{-1/2}[\phi_s + L\eta_s] + O(\phi_s^2 + \eta_s^2) \end{aligned} \tag{4.9}$$

where we have defined $K = K(s) \equiv 1 + h'(s)^2$ and $L = L(s) \equiv h'(s)$. With this notation we may rewrite (4.8) as

$$(x_s^2 + y_s^2)^{-1/2} = K^{-1/2} - K^{-3/2}\phi_s - K^{-3/2}L\eta_s + O(\phi_s^2 + \eta_s^2). \tag{4.10}$$

Turning to (2.13) and using (4.9), (4.10), and $\partial x/\partial s = 1 + \phi_s$, we find

$$\begin{aligned} \frac{\mathrm{T}(e, s)}{\sqrt{x_s^2 + y_s^2}}\frac{\partial x}{\partial s} &= \mathrm{T}(0, s)K^{-1/2} \\ &\quad + \phi_s\left\{(K^{-1/2} - K^{-3/2})\mathrm{T}(0, s) + \frac{\partial \mathrm{T}}{\partial e}(0, s)K^{-1}\right\} \\ &\quad + \eta_s\left\{\frac{\partial \mathrm{T}}{\partial e}(0, s)K^{-1}L - \mathrm{T}(0, s)K^{-3/2}L\right\} \\ &\quad + O(\phi_s^2 + \eta_s^2) \\ &\equiv \mu_1(s) + \mu_2(s)\phi_s + \mu_3(s)\eta_s + O(\phi_s^2 + \eta_s^2). \end{aligned} \tag{4.11}$$

Hence, combining (4.11) and (4.4), we approximate (2.13) by

$$\rho(s)\phi_{tt} = \frac{\partial}{\partial s}(\mu_1 + \mu_2\phi_s + \mu_3\eta_s) + \mu_4(s, t) + \mu_5(s, t)\phi + \mu_6(s, t)\eta \tag{4.12}$$

where

$$
\begin{aligned}
\mu_1(s) &\equiv \mathsf{T}(0, s)K^{-1/2}(s) \\
\mu_2(s) &\equiv \mathsf{T}(0, s)[K^{-1/2}(s) - K^{-3/2}(s)] + \frac{\partial \mathsf{T}}{\partial e}(0, s)K^{-1}(s) \\
\mu_3(s) &\equiv \frac{\partial \mathsf{T}}{\partial e}(0, s)K^{-1}(s)L(s) - \mathsf{T}(0, s)K^{-3/2}(s)L(s) \\
\mu_4(s, t) &\equiv \rho(s)F_1(s, h(s), t) \\
\mu_5(s, t) &\equiv \rho(s)\frac{\partial F_1}{\partial x}(s, h(s), t) \\
\mu_6(s, t) &\equiv \rho(s)\frac{\partial F_1}{\partial y}(s, h(s), t).
\end{aligned}
\tag{4.13}
$$

In a similar manner, using $\partial y/\partial s = h'(s) + \eta_s = L(s) + \eta_s$, we find

$$
\begin{aligned}
\frac{\mathsf{T}(e, s)}{\sqrt{x_s^2 + y_s^2}}\frac{\partial y}{\partial s} = {} & \mathsf{T}(0, s)K^{-1/2}L \\
& + \phi_s\left\{\frac{\partial \mathsf{T}}{\partial e}(0, s)K^{-1}L - \mathsf{T}(0, s)K^{-3/2}L\right\} \\
& + \eta_s\left\{(K^{-1/2} + L^2K^{-3/2})\mathsf{T}(0, s) + \frac{\partial \mathsf{T}}{\partial e}(0, s)K^{-1}L^2\right\} \\
& + O(\phi_s^2 + \eta_s^2).
\end{aligned}
\tag{4.14}
$$

Hence our approximation for (2.14) is given by

$$
\rho(s)\eta_{tt} = \frac{\partial}{\partial s}(\gamma_1 + \gamma_2\phi_s + \gamma_3\eta_s) + \gamma_4(s, t) + \gamma_5(s, t)\phi + \gamma_6(s, t)\eta \tag{4.15}
$$

where

$$
\begin{aligned}
\gamma_1(s) &\equiv \mathsf{T}(0, s)K^{-1/2}(s)L(s) \\
\gamma_2(s) &\equiv \frac{\partial \mathsf{T}}{\partial e}(0, s)K^{-1}(s)L(s) - \mathsf{T}(0, s)K^{-3/2}(s)L(s) \\
\gamma_3(s) &\equiv \mathsf{T}(0, s)[K^{-1/2}(s) + K^{-3/2}(s)L^2(s)] + \frac{\partial \mathsf{T}}{\partial e}(0, s)K^{-1}(s)L^2(s) \\
\gamma_4(s, t) &\equiv \rho(s)F_2(s, h(s), t) \\
\gamma_5(s, t) &\equiv \rho(s)\frac{\partial F_2}{\partial x}(s, h(s), t) \\
\gamma_6(s, t) &\equiv \rho(s)\frac{\partial F_2}{\partial y}(s, h(s), t).
\end{aligned}
\tag{4.16}
$$

For static problems, the model equations (4.12) and (4.15) take the form

$$\frac{\partial}{\partial s}\{(\mu_1, \mu_2, \mu_3)\cdot(1, \phi_s, \eta_s)\} + (\mu_4, \mu_5, \mu_6)\cdot(1, \phi, \eta) = 0 \tag{4.17}$$

$$\frac{\partial}{\partial s}\{(\gamma_1, \gamma_2, \gamma_3)\cdot(1, \phi_s, \eta_s)\} + (\gamma_4, \gamma_5, \gamma_6)\cdot(1, \phi, \eta) = 0. \tag{4.18}$$

We note that we thus have a coupled set of linear equations. Furthermore, even in the case where the tension function T is not dependent upon the spatial variables s (i.e., T $= \mathrm{T}(e)$), the coefficients μ_i, γ_i, $i = 1, 2, 3$ are spatially varying. Finally, we observe that in the event that $(\phi, n) = (0, 0)$ is a solution to (4.17) and (4.18), we have the relationships

$$\frac{\partial}{\partial s}(\mu_1(s)) + \mu_4(s, t) = 0 \tag{4.19}$$

$$\frac{\partial}{\partial s}(\gamma_1(s)) + \gamma_4(s, t) = 0 \tag{4.20}$$

which are conditions on T, E, and F in order that the equilibrium E be a steady state solution.

5. CONCLUDING REMARKS

Our discussions above have been limited to the planar motion of a one dimensional elastic body (i.e., a "string") for which we have derived, on the basis of fundamental laws of physics, equations for small amplitude motion about an equilibrium or reference position. Our emphasis has been on the differences between the equations for motion about a horizontal reference position and those for motion about a nonhorizontal, nonvertical reference position. As we have noted in the introduction, while the relationship between our considerations here and those needed for large flexible space structures (of more than one dimension) may appear somewhat tenuous, we believe that our findings are indicative of what one might expect in a full modeling attempt for flexible structures such as the Maypole Hoop/Column antenna mesh surface.

In summary we have the following conclusions.

(i) For a *horizontal* equilibrium or reference position, the small amplitude motion of a string with essentially transverse external forces is approximately described by a single scalar linear partial differential equation (the classifical wave equation in the absence of external forces).

(ii) For a *curved* (nonhorizontal) equilibrium or reference position, the small amplitude motion can be approximated by a coupled system of linear partial differential equations. The linearization assumption involves (see (H4.2)) the magnitudes of the slopes of the displacements *relative* to the equilibrium or reference position. Hence for "small amplitude motion", a linear model is probably adequate even for structures with substantial curvature in their reference positon. In general, the coupled system of equations cannot be reduced to a single equation unless additional nontrivial assumptions are made. Included in such assumptions are: (a) the elastic body is inextensible; or (b) the motion is essentially normal to the reference position at every point (or some other stringent assumption on the direction of the motion); or (c) the curved equilibrium position is essentially horizonal.

These findings suggest several rather important points in regard to modelling surfaces such as that depicted in Fig. 1. First, unless the equilibrium curvature of the surface is small, one should not expect small perturbations about the reference to be described (in the steady state) by a single Poisson equation

$$\Delta v + kv = f.$$

Rather, a system of equations will be required to describe the displacements. Secondly, since these equations will in general represent a linearization about a curved surface, measurements or observations of displacement should be made relative to the reference position. If the measurements are made in any other manner, nonlinearities may be important and it may be necessary to employ nonlinear partial differential equations to describe the motions.

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